Eliminating the parameters e and χ from the system (8.8) – (8.10), we arrive at the equations of the theory of plates.

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ON THE NORMALIZATION OF A HAMILTONIAN SYSTEM OF LINEAR DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS

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We construct an algorithm for seeking a real canonic transformation of a linear Hamiltonian system of differential equations to normal form. As an example we consider the application of this transformation in the restricted three-body problem.

1. We consider the Hamiltonian system of differential equations

$$dx / dt = IH(t)x, \qquad x = (x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n}) \tag{1.1}$$

The variables x_k and x_{n+k} are canonically conjugate (x_k are the coordinates, x_{n+k} are the momenta) in the corresponding mechanical problem. The $2n$ th-order symmetric matrix $H(t)$ is assumed real, continuous, 2π-periodic in t. The matrix I has the form

$$I = \begin{Vmatrix} 0 & E \\ -E & 0 \end{Vmatrix}, \qquad (I^{-1} = I' = -I, \; I^2 = -E, \; \det I = 1)$$

where E is the n th-order unit matrix.

The solution of a linear system is usually chosen as the generating solution when investigating stability, analyzing nonlinear oscillations, constructing approximate solutions of nonlinear Hamiltonian systems. Therefore, it is desirable to choose those coordinates in which the solution of the linear system (1.1) is described most simply.

System (1.1), as also every linear system with continuous periodic coefficients, is reducible [1]. This means that there exists a linear change of variables with a continuously differentiable matrix having a bounded inverse for all t and being such that system (1.1) is transformed into a system with constant coefficients. This change of variables is nonuniquely determined by system (1.1). Let the characteristic indices $\pm\ i\lambda_k$ $(k = 1,2, \ldots, n)$ of system (1.1) be purely imaginary, and let all the multipliers $\rho_k = \exp(i2\pi\lambda_k)$, $\rho_{n+k} = \bar{\rho}_k$ be distinct. Then, the most convenient coordinates are those in which the Hamiltonian function of the transformed system is described as a sum of the Hamiltonians of uncoupled oscillators

$$H = \frac{1}{2} \sum_{k=1}^{n} \lambda_k (y_k^2 + y_{n+k}^2) \tag{1.2}$$

We say that the corresponding system of differential equations has a normal form.

The problem of normalizing a linear Hamiltonian system with constant coefficients was investigated in sufficient detail in [2 – 11]. Normalization methods suitable for practical use were obtained in [9, 10]. The normalization of canonical systems with periodic coefficients were studied in [2, 12, 13]. The existence of a canonic transformation, 2π-periodic in t, normalizing system (1.1) was established in [2, 12]. It was shown in [12] that such a transformation can be obtained real. For $n = 1$ it was shown in [13] how to obtain in a practical manner a transformation normalizing system (1.1). Below we give a constructive method for setting up a linear canonic real transformation, 2π-periodic in t, of system (1.1) to normal form for an arbitrary n. The results are presented in such a way that they can be conveniently applied to solve concrete mechanical problems.

2. Let $\mathbf{X}(t)$ be the fundamental matrix – a solution of system (1.1), satisfying the condition $\mathbf{X}(0) = \mathbf{E}$. We represent the normalizing transformation $\mathbf{x} = \mathbf{N}\mathbf{y}$ as the succession of two changes of variables

$$\mathbf{x} = \mathbf{X}(t)\,\mathbf{A}e^{-\mathbf{B}t}\mathbf{z} \tag{2.1}$$

$$\mathbf{z} = \mathbf{C}\mathbf{y} \tag{2.2}$$

Here

$$\mathbf{B} = \left\| \begin{matrix} i\lambda_1 & & & & & \\ & \ddots & & & & \\ & & i\lambda_n & & & \\ & & & -i\lambda_1 & & \\ & & & & \ddots & \\ & & & & & -i\lambda_n \end{matrix} \right\|, \qquad \mathbf{C} = \left\| \begin{matrix} i\mathbf{E} & \mathbf{E} \\ -i\mathbf{E} & \mathbf{E} \end{matrix} \right\|$$

Transformation (2.1) takes system (1.1) to the diagonal form $d\mathbf{z} / dt = \mathbf{B}\mathbf{Z}$. After the application of transformation (2.2) the latter system of equations acquires the normal form with Hamiltonian function (1.2). In formula (2.1) we choose the matrix $\mathbf{A}$ such that the transformation $\mathbf{x} = \mathbf{N}\mathbf{y}$ is real, univalent, canonic, 2π-periodic in t. It can be verified that transformation (2.2) is canonic with valence $2i$. Furthermore, the matrices $\mathbf{X}(t)$ and $e^{-\mathbf{B}t}$ are simplicial since they are the solutions of Hamiltonian systems with initial conditions equal to the unit matrix. Indeed, let us verify, for example, the condition for the simpliciality of matrix $\mathbf{X}(t)$

$$\mathbf{X}'\mathbf{I}\mathbf{X} = \mathbf{I} \tag{2.3}$$

We compute the derivative of the left-hand side of equality (2.3). We obtain

$$\frac{d(\mathbf{X}'\mathbf{I}\mathbf{X})}{dt} = \frac{d\mathbf{X}'}{dt}\mathbf{I}\mathbf{X} + \mathbf{X}'\mathbf{I}\frac{d\mathbf{X}}{dt} = \mathbf{X}'\mathbf{H}'\mathbf{I}'\mathbf{I}\mathbf{X} + \mathbf{X}'\mathbf{I}\mathbf{H}\mathbf{X} \equiv 0$$

Consequently, the matrix $\mathbf{X}'\mathbf{I}\mathbf{X}$ is constant, but since it equals $\mathbf{I}$ for $t = 0$, equality (2.3) holds for all t. Thus, in order for the transformation $\mathbf{x} = \mathbf{N}\mathbf{y}$ to be canonic and univalent, it is necessary and sufficient [14] that $\mathbf{A}$ be a generalized simplicial matrix with valence $1/2i$, i.e. that the equality

$$\mathbf{A}'\mathbf{I}\mathbf{A} = \frac{1}{2i}\mathbf{I} \tag{2.4}$$

be fulfilled. Further, from the condition

$$\mathbf{X}(2\pi)\mathbf{A}e^{-2\pi\mathbf{B}}\mathbf{C} = \mathbf{X}(0)\mathbf{A}\mathbf{E}\mathbf{C}$$

for the 2π-periodicity of the normalizing transformation we obtain a matrix equation for determining $\mathbf{A}$,

$$\mathbf{X}(2\pi)\mathbf{A} = \mathbf{A}e^{2\pi\mathbf{B}}, \qquad e^{2\pi\mathbf{B}} = \left\| \begin{matrix} \rho_1 & & & & & \\ & \ddots & & & & \\ & & \rho_n & & & \\ & & & \bar\rho_1 & & \\ & & & & \ddots & \\ & & & & & \bar\rho_n \end{matrix} \right\| \tag{2.5}$$

The matrix $e^{2\pi\mathbf{B}}$ is the diagonal form of the matrix $\mathbf{X}(2\pi)$. The matrix $\mathbf{A}$, which reduces matrix $\mathbf{X}(2\pi)$ to diagonal form, as we see from Eq. (2.5), is constructed in the following manner [15]. Its columns must be the eigenvectors of the matrix $\mathbf{X}(2\pi)$. Namely, the jth column of matrix $\mathbf{A}$ is the eigenvector $\mathbf{e}_j$ of matrix $\mathbf{X}(2\pi)$, corresponding to the eigenvalue (multiplier) ρ_j. But since the eigenvectors are defined to within a scalar factor, the matrix – the solution of Eq. (2.5) – can be written in the form $\mathbf{A} = \mathbf{F}\mathbf{D}$, where $\mathbf{F}$ is some solution of Eq. (2.5) and $\mathbf{D}$ is a $2n$-order diagonal matrix whose elements are chosen so as to satisfy condition (2.4). Furthermore, we take it that the elements of matrix $\mathbf{D}$ are real numbers and that $d_{n+k,\,n+k} = d_{k,k}$, while the eigenvectors e_{n+k} and e_k are complex conjugate. This ensures the reality of the normalizing transformation.

3. Let us show how to find matrix $\mathbf{D}$. By substituting $\mathbf{A} = \mathbf{F}\mathbf{D}$ into equality (2.4) and taking into account that $\mathbf{D}' = \mathbf{D}$, we obtain

$$\mathbf{D}\mathbf{F}'\mathbf{I}\mathbf{F}\mathbf{D} = \frac{1}{2i}\mathbf{I} \tag{3.1}$$

We denote the matrix $\mathbf{F}'\mathbf{I}\mathbf{F}$ by $\mathbf{L}$. An element $l_{k,m}$ of this matrix equals the scalar product of the vectors $\mathbf{e}_k$ and $\mathbf{I}\mathbf{e}_m$

$$l_{k,m} = (\mathbf{e}_k \cdot \mathbf{I}\mathbf{e}_m)$$

But it can be verified that the equality

$$(\mathbf{u}\cdot\mathbf{I}\mathbf{v}) = -(\mathbf{I}\mathbf{u}\cdot\mathbf{v})$$

is valid for any vectors $\mathbf{u}, \mathbf{v}$. Consequently, matrix $\mathbf{L}$ is skew-symmetric. Let us investigate further the properties of matrix $\mathbf{L}$. We prove the following assertion.

Lemma. If the product of the eigenvalues ρ_k and ρ_m of a simplicial matrix $\mathbf{X}$ does not equal unity, the corresponding eigenvectors $\mathbf{e}_k$ and $\mathbf{e}_m$ satisfy the equality $(\mathbf{e}_k \cdot \mathbf{I}\mathbf{e}_m) = 0$.

Proof. By the definition of a simplicial matrix the equality

$$(\mathbf{IXu}\cdot\mathbf{Xv}) = (\mathbf{X}'\mathbf{IXu}\cdot\mathbf{v})$$

holds for any vectors $\mathbf{u}$ and $\mathbf{v}$. Using the simpliciality of matrix X, we obtain $(\mathbf{IXu}\cdot\mathbf{Xv}) = (\mathbf{Iu}\cdot\mathbf{v})$. Setting $\mathbf{u} = \mathbf{e}_m$ and $\mathbf{v} = \mathbf{e}_k$ in the latter equality, we obtain

$$(\mathbf{IXe}_m\cdot\mathbf{Xe}_k) = (\mathbf{Ie}_m\cdot\mathbf{e}_k) \tag{3.2}$$

But $\mathbf{Xe}_j = \rho_j\mathbf{e}_j$, therefore, equality (3.2) can be rewritten as:

$$(\rho_k\rho_m - 1)(\mathbf{e}_k\cdot\mathbf{Ie}_m) = 0$$

The lemma's assertion follows from the last equality.

The analysis carried out shows that matrix $\mathbf{L}$ has the form

$$\mathbf{L} = \begin{Vmatrix} 0 & \mathbf{M} \\ -\mathbf{M} & 0 \end{Vmatrix}$$

where $\mathbf{M}$ is an n-order diagonal matrix with elements $m_{kk} = (\mathbf{e}_k\cdot\mathbf{Ie}_{n+k})$. Not one of the elements m_{kk} can equal zero since otherwise the determinant of matrix $\mathbf{L}$ would equal zero. But

$$\det\mathbf{L} = \det\mathbf{F}'\det\mathbf{I}\det\mathbf{F} = (\det\mathbf{F})^2 \neq 0$$

since the matrix $\mathbf{F}$ is made up from the eigenvectors corresponding to distinct eigenvalues of matrix $\mathbf{X}(2\pi)$.

Let $\mathbf{r}_k$ and $\mathbf{s}_k$ be the real and imaginary parts of the eigenvector of matrix $\mathbf{X}(2\pi)$, corresponding to the eigenvalue ρ_k. Then, taking the complex conjugacy of vectors $\mathbf{e}_k$ and $\mathbf{e}_{n+k}$ into account, after simple manipulations we can obtain the expression

$$m_{kk} = -2i(\mathbf{r}_k\cdot\mathbf{Is}_k) \tag{3.3}$$

for the elements of matrix $\mathbf{M}$. From (2.4) and (3.3) we obtain an equation for finding d_{kk}

$$4d_{kk}^2(\mathbf{r}_k\cdot\mathbf{Is}_k) = 1 \tag{3.4}$$

The last equation has a real solution if the quantity $(\mathbf{r}_k\cdot\mathbf{Is}_k)$ is positive, which can always be achieved by an appropriate choice of the sign of λ_k in the Hamiltonian function (1.2). Indeed, by equating the real and the imaginary parts in the equation $\mathbf{Xe}_k = \rho_k\mathbf{e}_k$, we obtain a system of equations in $\mathbf{r}_k$ and $\mathbf{s}_k$,

$$\begin{gathered}(\mathbf{X} - \cos 2\pi\lambda_k\mathbf{E})\mathbf{r}_k + \sin 2\pi\lambda_k\mathbf{s}_k = 0 \\ -\sin 2\pi\lambda_k\mathbf{r}_k + (\mathbf{X} - \cos 2\pi\lambda_k\mathbf{E})\mathbf{s}_k = 0\end{gathered} \tag{3.5}$$

The system of Eqs. (3.5) does not alter under a simultaneous change of sign of λ_k and of the sign of the components of vector $\mathbf{r}_k$. Here, however, the sign of the scalar product $(\mathbf{r}_k\cdot\mathbf{Is}_k)$ does change to the opposite one. Thus, we have found the matrix $\mathbf{D}$. The matrix of the normalizing transformation $\mathbf{x} = \mathbf{Ny}$ has the form

$$\mathbf{N} = \mathbf{X}(t)\mathbf{FD}e^{-\mathbf{B}t}\mathbf{C}$$

After some manipulations it can be represented as a product of three real matrices

$$\mathbf{N} = \mathbf{X}(t)\mathbf{PQ}(t) \tag{3.6}$$

In formula (3.6) $\mathbf{P}$ denotes a constant matrix in which the kth column is the vector $-2d_{kk}\mathbf{s}_k$, and the $(n+k)$th column is the vector $2d_{kk}\mathbf{r}_k$ $(k = 1, 2, \ldots, n)$. The matrix $\mathbf{Q}(t)$ has the form

$$\mathbf{Q}(t) = \begin{Vmatrix} \cos\Lambda t & -\sin\Lambda t \\ \sin\Lambda t & \cos\Lambda t \end{Vmatrix}$$

$$\sin \Lambda t = \begin{Vmatrix} \sin \lambda_1 t & & \\ & \ddots & \\ & & \sin \lambda_n t \end{Vmatrix}, \qquad \cos \Lambda t = \begin{Vmatrix} \cos \lambda_1 t & & \\ & \ddots & \\ & & \cos \lambda_n t \end{Vmatrix}$$

4. As an example we find the transformation normalizing the system of linear equations which describe the motion in a neighborhood of a triangular libration point in the plane elliptic restricted three-body problem. In Nechvile coordinates with the true anomaly ν as the independent variable and for an appropriate choice of the unit of length, the motion is described by means of the Hamiltonian function [16]

$$H = \frac{1}{2}(p_1^2 + p_2^2) + p_1 q_2 - p_2 q_1 + \frac{1 - 4e\cos\nu}{8(1 + e\cos\nu)} q_1^2 + \frac{5 - 4e\cos\nu}{8(1 + e\cos\nu)} q_2^2 - \frac{3\sqrt{3}(1 - 2\mu)}{4(1 + e\cos\nu)} q_1 q_2$$

Let us take the parameters e and μ for the case of the Sun-Jupiter system: $e = 0.04825382$, $\mu = 0.00095388$. Computations on an electronic computer show that the solution matrix $\mathbf{X}(\nu)$ corresponding to the system of differential equation is, for $\nu = 2\pi$

$$\mathbf{X}(2\pi) = \begin{Vmatrix} 10.246067 & 15.765014 & -16.830551 & 9.400540 \\ -5.435207 & -8.372406 & 9.934193 & -5.646301 \\ 5.056440 & 8.591016 & -8.181647 & 5.105433 \\ 8.833277 & 15.135589 & -16.094789 & 10.055308 \end{Vmatrix}$$

The quantities λ_1, λ_2 are computed from the formulas [16]

$$\lambda_1 = 1 - \frac{1}{2\pi}\arccos\frac{a_1 + \Delta}{4}, \qquad \lambda_2 = -\frac{1}{2\pi}\arccos\frac{a_1 - \Delta}{4}$$

$$\Delta = (a_1^2 - 4a_2 + 8)^{1/2}$$

where a_1 is the trace of the matrix $X(2\pi)$, a_2 is the sum of all its principal second-order minors.

We obtain the numerical values $\lambda_1 = 0.996758$, $\lambda_2 = -0.080802$. We now need to find some solution of the system of Eqs. (3.5). For definiteness we assume the fourth components of the vector e_k real and equal to unity. The the real and imaginary parts of the eigenvectors are obtained as follows:

$$\mathbf{r}_1 = \begin{Vmatrix} 1.256976 \\ -1.371205 \\ -0.036985 \\ 1 \end{Vmatrix}, \qquad \mathbf{s}_1 = \begin{Vmatrix} 1.389429 \\ 0.273188 \\ 1.020730 \\ 0 \end{Vmatrix}$$

$$\mathbf{r}_2 = \begin{Vmatrix} 1.052220 \\ -0.607786 \\ 0.576385 \\ 1 \end{Vmatrix}, \qquad \mathbf{s}_2 = \begin{Vmatrix} -0.042113 \\ -0.040441 \\ -0.030937 \\ 0 \end{Vmatrix}$$

For the scalar products $(\mathbf{r}_k \cdot \mathbf{I}\mathbf{s}_k)$ we obtain $(\mathbf{r}_1 \cdot \mathbf{I}\mathbf{s}_1) = 1.061233$, $(\mathbf{r}_2 \cdot \mathbf{I}\mathbf{s}_2) = 0.032162$. Further, from Eqs. (3.4) we find the elements of matrix $\mathbf{D}$:

$$d_{11} = 0.485361, \qquad d_{22} = 2.788069$$

Now we can write out the normalizing matrix (3.6) in which

$$P = \begin{Vmatrix} -1.348748 & 0.234825 & 1.220173 & 5.867325 \\ -0.265189 & 0.225503 & -1.331058 & -3.389100 \\ -0.990844 & 0.172509 & -0.035902 & 3.214000 \\ 0 & 0 & 0.970721 & 5.576138 \end{Vmatrix}$$

The normalized system of differential equations is written in the form

$$dy / dv = \mathbf{IKy}, \qquad \mathbf{K} = \begin{Vmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_1 & \\ & & & \lambda_2 \end{Vmatrix}$$

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